Student seminar notes week 3

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1 Embeddings and units of number fields

Definition 1.1. Let F be a number field. An *embedding* is an injective homomorphism $F \hookrightarrow \mathbb{C}$.

Let F be a number field. By the primitive element theorem, it can be written as $F = \mathbb{Q}(\alpha)$ for some $\alpha \in F$. Consider the minimal polynomial m_α of α with roots $\alpha = \alpha_1, ..., \alpha_d$ in $\mathbb{C}$ (with $d = [F : \mathbb{Q}]$).
Since any field homomorphism φ verifies

$$m_\alpha(\varphi(\alpha)) = \varphi(m_\alpha(\alpha)) = \varphi(0) = 0,$$

the image $\varphi(\alpha)$ must be a conjugate of α.
So an embedding of F in $\mathbb{C}$ is entirely determined by the choice of the image $\varphi(\alpha)$ among the roots of m_α, and there are exactly d distincts embedding.

Now let r be the number of real embeddings and s be the number of pair of complex conjugate embeddings. We have that

$$[F : Q] = r + 2s$$

Remark 1.2. If F is a Galois extension of $\mathbb{Q}$, then we have $r = 0$ or $s = 0$ (since the Galois group cannot swap a real root and a pair of complex conjugate roots).

Theorem 1.3 (Dirichlet's unit theorem)**.** *Let F be a number field and r, s as above. We can find units $\varepsilon_1, ..., \varepsilon_{r+s-1} \in \mathcal{O}_F^\times$ such that*

$$\mathcal{O}_F^\times \cong \mathcal{W}_F \times \langle \varepsilon_1 \rangle \times \cdots \times \langle \varepsilon_{r+s-1} \rangle$$
$$\cong \mathcal{W}_F \times \mathbb{Z}^{r+s-1}$$

where $\mathcal{W}_F$ is the group of roots of unity in F. The ε_k are called a fundamental system of units *for F.*

Example 1.4. Let $F = Q[\sqrt{3}]$. We have two real embeddings, sending $\sqrt{3}$ to $\sqrt{3}, -\sqrt{3}$ respectively. So $r + s - 1 = 1$.
First, let us check that $2 + \sqrt{3}$ is an integer. Its minimal polynomial is

$$x^2 - 4x + 1 = (x - (2 + \sqrt{3}))(x - (2 - \sqrt{3}))$$

which is a monic polynomial with coefficients in $\mathbb{Z}$. Note that $(2+\sqrt{3})(2-\sqrt{3}) = 1$, so $2 + \sqrt{3} \in \mathcal{O}_F^\times$.
Since F is a subfield of $\mathbb{R}$, the group of roots of unity $\mathcal{W}_F$ is $\{\pm 1\}$. So by the Dirichlet unit theorem

$$\mathcal{O}_F^\times = \{\pm 1\} \times \langle 2 + \sqrt{3} \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}.$$

2 Completion of number fields

2.1 p-adic numbers

First let us recall a few definitions.

Definition 2.1. A *distance* on a field F is a function $d : F \times F \to [0, \infty[$ such that

- $d(x, y) = d(y, x) \quad \forall\ x, y \in F$,
- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, z) \leq d(x, y) + d(y, z) \quad \forall\ x, y, z \in F$ (the so-called *triangle inequality*).

Definition 2.2. A *norm* (also called an *absolute value*) on a field F is a function $|\cdot| : F \to [0, \infty[$ such that

- $|x| = 0$ if and only if $x = 0$,
- $|xy| = |x||y| \quad \forall\ x, y \in F$,
- $|x + y| \leq |x| + |y| \quad \forall\ x, y \in F$ (triangle inequality).

We say that a norm is *non-archimedean* when $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in F$. The non-archimedean inequality implies the triangle inequality, so it is a stronger property.

A norm $||\cdot||$ induces a distance $d_{||\cdot||}(x, y) := ||x - y||$.

Definition 2.3 (p-adic valuation)**.** Let $p \in \mathbb{Z}$ be a prime and $a \in \mathbb{Z} \setminus \{0\}$. We define $\text{ord}_p(a)$ to be the greatest integer n so that a can be written as

$$a = p^n b$$

with $b \in \mathbb{Z}$ not divisible by p.

We have that $\text{ord}_p(ab) = \text{ord}_p(a) + \text{ord}_p(b)$, like a logarithm.

Definition 2.4 (p-adic valuation for rationals)**.** Let $p \in \mathbb{Z}$ be a prime and $x = \frac{a}{b} \in \mathbb{Q}^\times$. We define

$$\mathrm{ord}_p(x) = \mathrm{ord}_p(a) - \mathrm{ord}_p(b)$$

Definition 2.5 (p-adic norm)**.** The p-adic valuation can be used to define a norm as follows :

$$|\cdot|_p : x \mapsto \begin{cases} 0 & \text{if } x = 0 \\ p^{-\mathrm{ord}_p(x)} & \text{otherwise} \end{cases}.$$

Example 2.6.

$$108 = 2^2 3^3$$

$$\mathrm{ord}_2(108) = 2 \quad \mathrm{ord}_3(108) = 3 \quad \mathrm{ord}_5(108) = 0$$

$$|108|_2 = 1/4 \quad |108|_3 = 1/21 \quad |108|_5 = 1$$

Proposition 2.7. *For all primes p, $|\cdot|_p$ is a non-archimedean norm.*

Proof : The first two conditions are easy to verify directly form the definition. We show that the non-archimedean inequality holds.
Let $x = \frac{a}{b}, y = \frac{c}{d} \in \mathbb{Q}$. We have $x + y = \frac{ad+bc}{bd}$. Without loss of generality, we may assume that $x, y, x + y \neq 0$.

$$\mathrm{ord}_p(x + y) = \mathrm{ord}_p(ad - bc) - \mathrm{ord}_p(bd)$$

$$\geq \min\{\mathrm{ord}_p(ad), \mathrm{ord}_p(bc)\} - \mathrm{ord}_p(b) - \mathrm{ord}_p(d)$$

$$= \min\{\mathrm{ord}_p(ad) - \mathrm{ord}_p(d) - ord_p(b), \mathrm{ord}_p(bc) - \mathrm{ord}_p(b) - \mathrm{ord}_p(d)\}$$

$$= \min\{\mathrm{ord}_p(a) - \mathrm{ord}_p(d), \mathrm{ord}_p(c) - \mathrm{ord}_p(b)\} = \min\{\mathrm{ord}_p(x), \mathrm{ord}_p(y)\}$$

$$\implies |x + y|_p \leq \max\{|x|_p, |y|_p\} \quad \square$$

Recall that two distances are equivalent if they have the same Cauchy sequences, and that two norms are equivalent if they induce equivalent distances.

Theorem 2.8 (Ostrowski)**.** *Every non-trivial norm of $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some prime p or for $p = \infty$.*

Let (a_n) and (b_n) be Cauchy sequences in $(\mathbb{Q}, |\cdot|)$. We say that that they have the same limit if the Cauchy sequence $(a_n - b_n)$ has limit 0.

Definition 2.9 (p-adic completion field)**.** The *field of p-adic numbers* $\mathbb{Q}_p$ is the *completion* of $\mathbb{Q}$ for the p-adic norm, meaning that it consists of all possible limits of Cauchy sequences in $(\mathbb{Q}, |\cdot|_p)$. We construct it as follows :

$$\mathbb{Q}_p := \{\text{Cauchy sequences in } (\mathbb{Q}, |\cdot|_p)\}/\sim$$

where $(a_n) \sim (b_n)$ if they have the same limit.

We can extend the p-adic norm to $\mathbb{Q}_p$ by letting it commute with the limit :

$$|x|_p = |\lim_i x_i|_p := \lim_i |x_i|_p.$$

Remark 2.10. We can take the completion for any norm on $\mathbb{Q}$ in a similar fashion. The completion of the standard norm $|\cdot|$ on $\mathbb{Q}$ is $\mathbb{R}$. Note that, while $|\cdot|$ on $\mathbb{Q}$ takes values in $\mathbb{Q}$, the extended norm $|\cdot|$ on $\mathbb{R}$ takes values in $\mathbb{R}$. In the case of a p-adic norm however, the extended norm $|\cdot|_p$ still takes values in $\mathbb{Q}$. Indeed, it only takes countably many values $\{p^{-n} \mid n \in \mathbb{N}\} \cup \{0\}$, so the only accumulation point is 0.

Remark 2.11. Directly from the definition, $\mathbb{Q}_p$ is complete for the p-adic norm. $\mathbb{Q}_p$ is a field extension of $\mathbb{Q}$, but not a finite extension. In fact, $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ (as any number in $\mathbb{Q}_p$ can be approached by a sequence in $\mathbb{Q}$) but we do not prove it.

Definition 2.12 (p-adic integer)**.** The *ring of p-adic integers* is

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}.$$

Theorem 2.13.

$$\mathbb{Z}_p \approxeq \varprojlim_{p \to \infty} \mathbb{Z}/p^n\mathbb{Z}$$

which means that for $x \in \mathbb{Z}_p$, we can find a unique sequence of integers $(x_i)_{i \in \mathbb{N}}$ such that for all i

$$0 \leq x_i \leq p^i \ ; \qquad x_i \equiv x_{i+1} \pmod{p^i}.$$

Let we write x_i, x_{i+1} in base p :

$$x_i = b_0 + b_1 p + b_2 p^2 + \cdots + b_{i-1} p^{i-1}$$

$$x_{i+1} = c_0 + c_1 p + c_2 p^2 + \cdots + c_i p^i$$

(with the digits $0 \leq b_j, c_k \leq p-1$)
The second condition given by Theorem 2.13 implies that for all $j \leq i-1$, $b_j = c_j$, so we can think of x as an infinite p-adic expansion

$$x = \sum_{j=0}^{\infty} b_j \ (p^j)$$

We also have a p-adic expansion for all elements of $\mathbb{Q}_p$, since for all $y \in \mathbb{Q}_p$, there exists some n such that $p^n y$ is a p-adic integer. So we can write the expansion of $p^n y$ and "shifts the digits" n places to the left.

Example 2.14. The p-adic expansion of 1 is

$$1 = 1 + 0 \ (p) + 0 \ (p^2) + \cdots$$

The p-adic expansion of -1 is

$$-1 = p - 1 + p - 1 \ (p) + p - 1 \ (p^2) + \cdots$$

We can verify that

$$1 + \sum_{j=0}^{\infty} (p-1) p^j = 0$$

Example 2.15. Let us find the 5-adic expansion of $\frac{1}{3}$.
We write
$$\frac{1}{3} = a_0 + a_1\ (5) + a_2\ (25) + \cdots$$
which translates into the set of conditions :

- $3a_0 \equiv 1 \pmod 5 \implies a_0 = 2$
- $3(2 + 5a_1) \equiv 1 \pmod{25} \implies 15a_1 \equiv 20 \pmod{25}$
$\implies 3a_1 \equiv 4 \pmod 5 \implies a_1 = 3$
- similarly for $a_2 = 3, a_3 = 1$, etc.

Eventually we find a sequence of digits that repeat. We can guess a pattern, in this case
$$\frac{1}{3} = 2313131313\cdots = 2\overline{31}$$
and then check through direct computation that indeed $3 \times 2\overline{31} = 1$.

2.2 Completion of a general number field

Let F be any number field. We will define a (non-archimedean) norm on F using a prime ideal of $\mathcal{O}_F$, and then use the completion with respect to this norm. This construction preserves some algebraic properties of the number field, and we allow us to study it "locally" in some sense.

Let $\mathfrak{p}$ be an ideal in $\mathcal{O}_F$.

Definition 2.16 ($\mathfrak{p}$-adic valuation, $\mathfrak{p}$-adic norm)**.** Let $x \in F^\times$. We have a prime ideal decomposition
$$x\mathcal{O}_F = \mathfrak{p}^n \mathfrak{p}_1^{n_1} ... \mathfrak{p}_t^{n_t}$$
where the $\mathfrak{p}_i$ are distinct from $\mathfrak{p}$. We set $\mathrm{ord}_\mathfrak{p}(x) := n$ (the highest power of $\mathfrak{p}$ appearing in the decomposition). The $\mathfrak{p}$-*adic norm* on F is
$$|\cdot|_\mathfrak{p} : x \mapsto \begin{cases} 0 & \text{if } x = 0 \\ c^{\mathrm{ord}_\mathfrak{p}(x)} & \text{otherwise} \end{cases}.$$
for some real number $0 < c < 1$. Different choices of c yield equivalent norms.

Remark 2.17. In the previous section we had the particular case
$$\mathcal{O}_F = \mathbb{Z}, \quad \mathbb{Z} \cap \mathfrak{p} = p\mathbb{Z}, \quad c = \frac{1}{p}.$$
We leave the proof that this is a non-archimedean norm as an exercise.
In the archimedean case, Let $\sigma : F \to \mathbb{C}$ be an embedding. Then the pullback norm of the usual norm on $\mathbb{C}$
$$|x|_\sigma = |\sigma(x)|_C$$
is an archimedean norm on F.

Remark 2.18. If $\sigma_1, ..., \sigma_r$ are the real embeddings of F and $\sigma_{r+1}, ..., \sigma_{r+s}$ representants of conjugate pairs of complex embeddings, then :

- the $|\cdot|_{\sigma_i}$ are pairwise inequivalent
- $|\cdot|_\sigma$ and $|\cdot|_{\bar{\sigma}}$ are equivalent

Furthermore, for $\mathfrak{p} \neq \mathfrak{q}$, the norms $|\cdot|_\mathfrak{p}$ and $|\cdot|_\mathfrak{q}$ are inequivalent (see exercise sheet).

Example 2.19. Let $F = \mathbb{Q}[x]/(x^2-3)$. Let $\rho, -\rho$ be the roots of 3 in F. We have two non-equivalent absolute values on F:

$$\begin{cases} |a+b\rho|_1 = |a+b\sqrt{3}|_\mathbb{R} & (\sigma_1 : \rho \mapsto \sqrt{3} \in \mathbb{R}) \\ |a+b\rho|_2 = |a-b\sqrt{3}|_\mathbb{R} & (\sigma_2 : \rho \mapsto -\sqrt{3} \in \mathbb{R}) \end{cases}$$

Theorem 2.20 (Ostrowski)**.** *Every non-trivial norm on F is either equivalent to a $\mathfrak{p}$-adic norm for some $\mathfrak{p}$ (if non-archimedean) or to the pullback norm of some embedding $F \to \mathbb{C}$ (if archimedean).*

As before, two Cauchy sequences has the same limit if their difference has limit 0.

Definition 2.21. The completion of F with respect to a norm $||\cdot||$ is the field

$$\{\text{Cauchy sequences in } (F, ||\cdot||)\}/\sim$$

where $(a_n) \sim (b_n)$ if they have the same limit.

The norm $||\cdot||$ extends to the completion by commuting with the limit :

$$||\lim_i x_i|| = \lim_i ||x_i||.$$

Notation :

- if $||\cdot||$ is the $\mathfrak{p}$-adic norm $|\cdot|_\mathfrak{p}$, we will denote the completion $F_\mathfrak{p}$.
- if $||\cdot||$ is an archimedean norm given by an embedding, then the completion will be isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ (via an isometry for the induced distance).

Definition 2.22 ($\mathfrak{p}$-adic ring of integers)**.** The *ring of $\mathfrak{p}$-adic integers* $\mathcal{O}_\mathfrak{p}$ is

$$\{x \in F_\mathfrak{p} \mid |x|_\mathfrak{p} \leq 1\}$$

$\mathcal{O}_\mathfrak{p}$ is a local ring. Its group of units is

$$\mathcal{O}_\mathfrak{p}^\times = \{x \in F_\mathfrak{p} \mid |x|_\mathfrak{p} = 1\}$$

and

$$\mathfrak{m}_\mathfrak{p} = \mathcal{O}_\mathfrak{p} \setminus \mathcal{O}_\mathfrak{p}^\times$$

is the unique maximal ideal (we have that $\mathfrak{m}_\mathfrak{p} = \mathfrak{p}\mathcal{O}_\mathfrak{p}$).

Remark 2.23. We can define the residue field of $F_\mathfrak{p}$ as $\mathcal{O}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$. The residue field of $F_\mathfrak{p}$ is isomorphic to the residue field of F, so it is often denoted by $\mathbb{F}_\mathfrak{p}$:

$$\mathcal{O}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p} \cong \mathcal{O}_F/\mathfrak{p} =: \mathbb{F}_\mathfrak{p},$$

so it is finite and $F_\mathfrak{p}$ is local (complete with finite residue field). In fact, $\mathcal{O}_\mathfrak{p} \cap F$ is exactly the localization of $\mathcal{O}_F$ at the ideal $\mathfrak{p}$.

Proposition 2.24 ($\mathfrak{p}$-adic expansion). *Let $\mathcal{C} = \{c_0 = 0, c_1, \cdots, c_q\}$ be a set of representatives of $\mathcal{O}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$ and let $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$. We call π a uniformizer of $F_\mathfrak{p}$.*

- *$\forall\ x \in \mathcal{O}_\mathfrak{p}$, we have*

$$x = \sum_{j=0}^{\infty} a_j \pi^j, \quad (a_j \in \mathcal{C} \quad \forall\ j).$$

- *$\forall\ x \in F_\mathfrak{p}$ we have*

$$x = \sum_{j=-\,ord_\mathfrak{p}(x)}^{\infty} a_j \pi^j \quad (a_j \in \mathcal{C} \quad \forall\ j).$$

- $\mathfrak{m}_\mathfrak{p} = \langle \pi \rangle$.

(Idea of a proof : Let $x \in \mathcal{O}_\mathfrak{p}$, take a_0 a rep of the class $x + \mathfrak{m}_\mathfrak{p}$, then set $x_1 = \frac{x - a_0}{\pi} \in \mathcal{O}_\mathfrak{p}$. Iterate the process.)

Theorem 2.25 (Hensel's lemma). *Let f be a monic polynomial in $O_\mathfrak{p}[x]$ and$\bar{f}$ be its class in the residue field $\mathbb{F}_\mathfrak{p}[x] = \mathcal{O}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}[x]$.*
If $\bar{g}(x)$ and $\bar{h}(x)$ are monic and coprime in the residue field such that $\bar{f} = \bar{g}\bar{h}$ then there exists a lifting of $\bar{g}$ and $\bar{h}$ to polynomials $g, h \in \mathcal{O}_\mathfrak{p}[x]$ such that $f = gh$ that preserves their degree (i.e. $\deg(g) = \deg(\bar{g}), \deg(h) = \deg(\bar{h})$).

For a proof, see Theorem 4.6 in [3].
In particular, whenever one of the factors $\bar{g}, \bar{h}$ is linear, we can find a root of f in $\mathcal{O}_\mathfrak{p}[x]$.

Example 2.26. Let $f(x) = x^2 - 2 \in \mathbb{Z}_7[x]$. Then in $\mathbb{F}_7[x]$, $\bar{f}$ factors as

$$\bar{f}(x) = (x - 3)(x - 4)$$

So $\exists\ g, h \in \mathbb{Z}_7[x]$ monic linear such that $f(x) = g(x)h(x)$, and therefore 2 has a square root in $\mathbb{Z}_7$.

3 Ramification theory

Take K/F an extension of number fields, $\mathfrak{p}$ a prime ideal in $\mathcal{O}_F$ and consider the prime decomposition

$$\mathfrak{p}\mathcal{O}_K = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}\cdots\mathfrak{P}_g^{e_g}.$$

where the $\mathfrak{P}_j$ are distincts primes in $\mathcal{O}_K$ and the e_j are positive integers.

The norms $|\cdot|_{\mathfrak{P}_i}$ on K are all pairwise inequivalent, but they all restrict to $|\cdot|_{\mathfrak{p}}$ up to equivalence. In fact, the $\mathfrak{P}_i$ represent all the topologically inequivalent ways to extend $|\cdot|_{\mathfrak{p}}$ to K. At the completion level, we have the following finite extension for each $\mathfrak{P}_i$:

$$K_{\mathfrak{P}_i} / F_{\mathfrak{p}}$$

and conversely if $L/F_{\mathfrak{p}}$ is a finite extension, then $L = K_{\mathfrak{P}}$ for some finite extension K/F and some prime ideal $\mathfrak{P}$ in $\mathcal{O}_K$. This is a way of saying that any finite extension of $F_{\mathfrak{p}}$ can be obtained from a prime ideal in a finite extension of F.

For $K_{\mathfrak{P}} / F_{\mathfrak{p}}$, let $\mathcal{O}_{\mathfrak{P}}$ be the ring of integer of $K_{\mathfrak{P}}$. It has a unique maxiaml ideal $\mathfrak{m}_{\mathfrak{p}}\mathcal{O}_{\mathfrak{P}}$. As with number fields, we have a unique factorization

$$\mathfrak{m}_{\mathfrak{p}}\mathcal{O}_{\mathfrak{P}} = \mathfrak{p}\mathcal{O}_{\mathfrak{P}} = \mathfrak{m}_{\mathfrak{P}}^e$$

where $e = e(\mathfrak{m}_{\mathfrak{P}} / \mathfrak{m}_{\mathfrak{p}})$ is the *local ramification index.*
We also have the *local residue field degree* :

$$f = f\left(\mathfrak{m}_{\mathfrak{P}} / \mathfrak{m}_{\mathfrak{p}}\right) = \left[\mathcal{O}_{\mathfrak{P}} / \mathfrak{m}_{\mathfrak{P}} ; \mathcal{O}_{\mathfrak{p}} / \mathfrak{m}_{\mathfrak{p}}\right]$$

Remark 3.1. In this case, there is a single prime in the factorization of $\mathfrak{m}_{\mathfrak{p}}\mathcal{O}_{\mathfrak{P}}$ because we "killed" all the other primes by localizing.

Definition 3.2. We say that the extension $K_{\mathfrak{P}} / F_{\mathfrak{p}}$ is

- ramified if $e > 1$.
- unramified if $e = 1$.
- totally ramified if $f = 1$.

Remark 3.3. Note that

$$f\left(\mathfrak{m}_{\mathfrak{P}} / \mathfrak{m}_{\mathfrak{p}}\right) = \left[\mathcal{O}_{\mathfrak{P}} / \mathfrak{m}_{\mathfrak{P}} ; \mathcal{O}_{\mathfrak{p}} / \mathfrak{m}_{\mathfrak{p}}\right] = [\mathbb{F}_{\mathfrak{P}} ; \mathbb{F}_{\mathfrak{p}}] = f\left(\mathfrak{P} / \mathfrak{p}\right).$$

Similarly we can check that $e\left(\mathfrak{m}_{\mathfrak{P}} / \mathfrak{m}_{\mathfrak{p}}\right) = e\left(\mathfrak{P} / \mathfrak{p}\right)$.
So the local ramification index and residue field degree (for the completions $K_{\mathfrak{P}}, F_{\mathfrak{p}}$) are the same as their global counterpart (for K, F).

If K/F is Galois, then the local extension ${}^{K_{\mathfrak{P}}}\!/\!_{F_{\mathfrak{p}}}$ is also Galois, but since there is no splitting (because there is only one prime, see remark 3.1), we have that

$$ef = [K_{\mathfrak{P}} : F_{\mathfrak{p}}]$$

(compare this formula with the non-local case seen in Week 2: $efg = [K : F]$). This means we have a simpler layer decomposition

$$F_{\mathfrak{p}} \xhookrightarrow{f} K_{\mathfrak{P}}^I \xhookrightarrow{e} K_{\mathfrak{P}}$$

where ${}^{K_{\mathfrak{P}}^I}\!/\!_{F_{\mathfrak{p}}}$ is unramified, and ${}^{K_{\mathfrak{P}}}\!/\!_{K_{\mathfrak{P}}^I}$ is totally ramified (see *Normal extensions* in [4]).

3.1 Finite extensions of $\mathbb{Q}_p$

$\mathbb{Q}_p$ is not algebraically closed, and its closure $\mathbb{Q}_p^{alg}$ is not complete. However, the completion of $\mathbb{Q}_p^{alg}$ is both complete and algebraically closed. We will denote it $\mathbb{C}_p$. There is a unique absolute value on $\mathbb{C}_p$ that extends $|\cdot|_p$ and we also denote it $|\cdot|_p$.
Suppose that $\sigma : F \hookrightarrow \mathbb{Q}_p^{alg}$ is an embedding. For $x \in F$, we define :

$$|x|_\sigma = |\sigma(x)|_p$$

so that $|\cdot|_\sigma$ extends $|\cdot|_p$ on F.
Moreover, we have a one-to-one correspondance :

$$\left\{\text{topologically distinct extensions of } |\cdot|_p \text{ to } F\right\}$$

$$\updownarrow$$

$$\left\{\text{conjugacy classes of continuous embeddings } \ \sigma : F \hookrightarrow \mathbb{Q}_p^{alg}\right\}$$

So the only way to extend an absolute value from $\mathbb{Q}_p$ to F is via an embedding.

Sources

- [1] Nancy Childress, *Class Field Theory*
- [2] Neal Koblitz, *p-adic Numbers, p-adic Analysis and Zeta-functions*
- [3] Jürgen Neukrich, *Algebraic Number Theory*
- [4] Frédérique Oggier, *Introduction to Algebraic Number Theory*